Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-12-03

Warm-up

Question 1. T/F: The gradient $\nabla f(a, b)$ gives a normal vector to the surface $z = f(x, y)$ at the point where $x = a$ and $y = b$.

Question 2. T/F: If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}$ is always perpendicular to $\mathbf{r}'$.

From last time

Problem 1. Let $\mathbf{F} = \langle a, b, c\rangle$ where a, b, c are constants. Let D be the region $x^2 + y^2 + z^2 \le 1$ and let E be the solid cube $-2 \le x, y, z \le 2$.

(a) Compute $\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS$ and $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS$.
(b) Compute $\iint_{\partial D} |\mathbf{F} \cdot \mathbf{n}|\, dS$ and $\iint_{\partial E} |\mathbf{F} \cdot \mathbf{n}|\, dS$.

As usual, we orient ∂D and ∂E outwards.

Problem 2. Let $\mathbf{r}(t)$ parametrize the space curve C, with $|\mathbf{r}'(t)| = 1$ for all t.

(a) If $f(x, y, z)$ is a function on $\mathbb{R}^3$, show that
$$\frac{d}{dt}(f(\mathbf{r}(t))) = D_{\mathbf{r}'(t)} f.$$
(b) Suppose that $\mathbf{r}(t)$ parametrizes the curve of intersection of $2z = x + y + 3$ and $z^2 = x^2 + y^2$ with $\mathbf{r}(0) = (3, 4, 5)$ and $|\mathbf{r}'(0)| = 1$, counterclockwise when viewed from above. Find dz/dt at $t = 0$.

Miscellaneous problems

Problem 1. Find the angle between the two planes $x+y+2z = 3$ and $x + y - z = 1$.

Problem 2. Write a Cartesian equation for the tangent line to the parametric curve $x = t^3 - 12t$, $y = t^2$ at the point $(-16, 4)$.

Problem 3. Find a linear approximation to the function $f(x, y) = e^{x+y} \sin(x + y)$ at the point $(x, y) = (0, 0)$.

Problem 4. Consider the vector field
$$\mathbf{F} = \left\langle \frac{1}{x^2} + y, x + \frac{1}{y} + z^2, 2zy \right\rangle.$$
If it is conservative, find a potential function f such that $\nabla f = \mathbf{F}$. If it is not conservative, explain why no such f can exist.

Problem 5. Find the maximum value of the function $f(x, y) = x^3 + y^3$ in the region $x^4 + y^4 \le 1$.

Problem 6. Find the length of the polar curve $r = \theta^2 - 1$, $1 \le \theta \le 2$.

Problem 7. At the point $(x, y) = (2\pi, 0)$ on the polar curve $r = \theta \cos\theta$, find the slope dy/dx.

Problem 8. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle \sin x, \cos y, xz\rangle$ and C is parametrized as $\mathbf{r}(t) = \langle t^3, -t^2, t\rangle$, $0 \le t \le 1$.

Problem 9. Suppose that f, g are defined on $\mathbb{R}^3$ and C is any closed curve. Show that
$$\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0.$$
What if f, g are not defined on all of $\mathbb{R}^3$?

Problem 10. Show that $f(x, y) = \cos(x + y)$ has a critical point at $(0, 0)$, and classify it as either a local min, local max, or saddle point.

Problem 11. Find the area of the part of the plane $x+2y+3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$.

Problem 12. Let $\mathbf{a}, \mathbf{b}$ be vectors in $\mathbb{R}^3$, and suppose that $\mathbf{a} + \mathbf{b}$ is orthogonal to $\mathbf{a} - \mathbf{b}$. Show that $|\mathbf{a}| = |\mathbf{b}|$.

Problem 13. Compute the double integral
$$\int_0^1 \int_0^{\cos^{-1}(y)} \sin(\sin(x))\, dx\, dy.$$

Problem 14. Let $f(x, y, z) = \sin(x + y\tan(x)) + \cos(x + z\tan(z)) + \ln(y + z\tan(y))$. Compute f_{xyz}.

Problem 15. The volume enclosed by a sphere of radius R is $\frac{4}{3}\pi R^3$. Use this fact, and an appropriate change of variables, to deduce a formula for the volume enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a, b, c > 0$.

Problem 16. Set up, but do not evaluate, a triple integral giving the volume of the bounded region between the planes $x + y + z = 1$, $x - y + z = 1$, $x = 0$, and $z = 0$.

Problem 17. Let f be a function of x, y, and let r, θ denote polar coordinates for the xy-plane. Assume that $r \ge 0$. Suppose that at the point $(6, 8)$, we have $\partial f/\partial r = 4$ and $\partial f/\partial\theta = -2$. Find $f_x(6, 8)$ and $f_y(6, 8)$.

Problem 18. A lamina occupies the region of the xy-plane inside $x^2 - 2x + y^2 = 0$ but outside $x^2 + y^2 = 1$, and has density $\sigma(x, y) = |y|$. Find its center of mass.

Problem 19. Observe that $1^7 - (1)(1)^6 + (2)(1) - 2 = 0$. Now use implicit differentiation to approximate a solution to
$$x^7 - 1.03x^6 + 2.06x - 2 = 0.$$
(Hint: consider the equation $x^7 - ax^6 + bx - 2 = 0$ as implicitly defining x in terms of a, b near $(a, b, x) = (1, 2, 1)$.)

Problem 20. Let D be the region lying between the lines $y = 2x + 1$, $y = 2x + 4$, $y = -3x + 1$, $y = -3x + 4$. Find

$$\iint_D \frac{y - 2x}{y + 3x}\,\mathrm{d}x\,\mathrm{d}y.$$

Problem 21. Let C be a circle of radius 1 in the plane $x+2y+z = 4$ centered at the point $(1, 2, -1)$ and oriented clockwise when viewed **from the origin**. Find

$$\int_C y\,\mathrm{d}x + 2x\,\mathrm{d}y + (2x - y)\,\mathrm{d}z.$$

Problem 22. Suppose that $x^{1/3}y^{2/3} = 1$, $x \geq 0$, $y \geq 0$.

(a) Find the values of x and y for which

$$ux + vy$$

is minimal, in terms of fixed $(u, v) \in \mathbb{R}^2$.

(b) Compute the Jacobian determinant

$$\frac{\partial(x, y)}{\partial(u, v)}.$$

Could you have figured this out without solving the preceding part?

Problem 23. Let $h(t) = \dfrac{t}{\sqrt{1 + t^2}}$, and suppose that $u(x, y)$ satisfies the functional equation

$$u - h(x - yu) = 0.$$

(Note that $h(x - yu)$ means the function h applied to the input $x - yu$, not multiplication.) Show that u then satisfies the partial differential equation

$$\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = 0.$$

Problem 24. Let S be a sphere of radius R centered at the origin, and suppose a, h are such that

$$-R \leq a \leq a + h \leq R.$$

Evaluate the surface area of the portion of S that lies between the planes $z = a$ and $z = a + h$, and show that it does not depend on a; only on h.

Problem 25. Let S be given in spherical coordinates by an equation $\rho = f(\theta)$, where $(\theta, \phi) \in D$. Show that the area of S is

$$\iint_D \sqrt{\sin^2\phi f(\theta)^4 + f(\theta)^2 f'(\theta)^2}\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

(This is mostly just tedious.)

Problem 26. Find the flux of the vector field

$$\langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle$$

through the upper hemisphere ($z \geq 0$) of the unit sphere $x^2 + y^2 + z^2 = 1$, oriented upwards.

Problem 27. Find the volume of the region described by the two simultaneous inequalities

$$(x - 2)^2 + y^2 + z^2 \leq 4$$
$$x^2 + y^2 + (z - 2)^2 \leq 4.$$

You may find it helpful to exploit symmetry.